A MATHEMATICAL STUDY OF ARBITRAGE

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SUMMARY

This paper is a systematic study of the mathematical structure underlying nearly perfect exchange markets which are spatially or temporally separated. The principal questions investigated are "What are equilibrium conditions for a set of exchange rates?" and "How can arbitrage possibilities be discovered, if they exist?" The analysis involves the combined use of an algebraic representation, which is conducive to the derivation of qualitative features characterizing a multi-exchange market; and two linear programming models, one of which has use in establishing a desirable set of equilibrium exchange rates, and the other of which has a special form permitting an efficient computational scheme for discovering arbitrage possibilities.

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INTRODUCTION

Arbitrage is an important equilibrating mechanism in all
nearly perfect exchange markets which are spatially or temporally
separated. For example, a differential in foreign exchange rates
which allows the possibility of buying a money in one market and
selling it at a net gain in another is soon removed by the action
of arbitragers; the profitable transactions, by creating addi-
tional demand for one currency and supply of the other, drive the
foreign exchange rates back to equilibrium. Similar economic
forces are present where trading takes place in stocks and shares,
bullion, marine insurance, and commodities for spot and future
delivery. This paper is a systematic study of the mathematical
structure underlying such markets. We primarily address ourselves
to the questions, "What are equilibrium conditions for a set of
exchange rates?" and "How can arbitrage possibilities be dis-
covered, if they exist?"

Our analysis involves the combined use of an algebraic repre-
sentation, in Section 1, which is conducive to the derivation of
qualitative features characterizing a multi-exchange market, and
of two linear programming models, Sections 2 and 3.[*] One of these
linear programming models has use in establishing a "best" set
of equilibrium exchange rates (the definition of "best" is given
below) while the other has a special form permitting an efficient

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[*]Some of our results are derivable using either one of the
techniques of analysis; in such cases, we have attempted to employ
the method which appears most immediate.

computational scheme for discovering arbitrage transactions.

We begin by considering only n—country <u>currency</u> exchange. Section 1.1 is devoted to situations of "pure exchange" markets or networks in which buying and selling rates between any two currencies are exact reciprocals (thus ruling out two—currency arbitrage) and do not differ in the associated two countries.[*] Section 1.2 is concerned with "general exchange networks," in which two—currency arbitrage may exist, brokerage fees may be levied against transactions, dealings in commodities, bullion, stocks, etc. are permitted as exchange possibilities, and exchange rates may not exist explicitly for all pairs of currencies and commodities. Sections 2 and 3 contain the programming models which solve for equilibrium systems and arbitraging schemes, respectively.

1. AN ALGEBRAIC REPRESENTATION OF EXCHANGE TRANSACTIONS

Definitions and rules of operation

By a <u>network</u> we shall mean a set of countries or, equivalently, currencies and a set of rates between pairs of countries. The countries are thought of as nodes and the rates as arcs on a graph. If a rate is prescribed between countries X_1 and X_j , the rate between X_j and X_1 will also be prescribed;[**] in other words, it is assumed that whenever a market exists for the purchase of one currency in exchange for another, there is

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[*]For example, we assume the exchange rate of dollars for pounds and of pounds for dollars are exact reciprocals of each other, and the dollar—to—pound rate is the same in the United States as in Britain.

[**]Throughout the paper, we assume that no rate is zero.

concomitantly a market for the opposite exchange.[*] If these
rates are always reciprocal, the network is called a _pure ex-
change_ network. If rates are prescribed between all pairs of
countries, the network is designated a _complete_ network. Networks
which are not complete are referred to as _incomplete_ networks.[**]
We use the term "currency devaluation" in a broad sense, not
distinguishing between devaluation and appreciation. We shall
always mean by devaluation the action of a country in changing
its rates by a constant factor with respect to _all_ other countries.
We define "$X_i X_j$" to be the rate of exchange of country X_i's
currency for that of X_j's. A series of letters, e.g. $X_2 X_7 X_3 X_4 X_5 X_7$,
is defined to be the product of the numbers $X_2 X_7$, $X_7 X_3$, $X_3 X_4$,
$X_4 X_5$, $X_5 X_7$. Such a series of letters, to be referred to as a
chain, is seen to represent the number of units of the last country's
money which might be obtained by taking one unit of the first
country's money and sending it through the indicated series of
countries.[***] In many cases we must discuss general kinds of
chains, and accordingly, for transactions of secondary interest
in the computation, we do not indicate the exact country involved
but simply number the countries by a superscript. For example,
$X_1 X_3 X^1 X^2 X^3 X^4 X^5 X_1$, indicates money flowing from country X_1 to X_3 and

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[*]Such a condition might not hold for a financier in a country
imposing tight exchange control; the central bank might be willing
to exchange domestic currency for some scarce currency but unwilling
to make the opposite transaction.

[**]We assume that all networks are connected, i.e., it is
possible to exchange any currency for any other currency via some
series of transactions

[***]A string of letters may represent either a number or a
series of countries. Context will make our usage clear.

going through five other countries, (not necessarily distinct),
before returning to X_1 . Chains which have the property of
the above example, that they begin and end with the same letter,
are called cycles. To arbitrage in a currency network is to per-
form a series of exchange transactions resulting in no net loss
of any currency and a net gain in some currency. Examples of
arbitrage might involve a "cycle" of transactions such as trading
dollars for pounds, pounds for frances, francs for lire, lire for
dollars, or a simultaneous exchange of lire for pounds, pounds for
francs, francs for lire, with profit in lire which is then trans-
formed into dollars. In particular, a cycle permits arbitrage
when its value is greater than 1. There can be no arbitrage
in a network unless such a cycle exists.[*] A network is said to
be in _equilibrium_ if it is not possible to arbitrage in that
network.

The above definitions are seen to define the following rule:[**]

(1) $$X_1 X^1 \ldots X^p X_j X^1 \ldots X^q X_k = X_1 X^1 \ldots X^p X_j \cdot X_j X^1 \ldots X^q X_k .$$

Our definition of a pure exchange network gives

(2) $$X_i X_j X_i = 1 \text{ for all } i, j .$$

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[*] If a chain does not return to its point of origin and "buy"
at least as much as it "sold" when it began, the arbitrager would
be in debt at that point.

[**] A raised dot between series of letters indicates multiplication.

1.1 COMPLETE PURE EXCHANGE NETWORKS

Most elementary expositions of arbitrage illustrate such
schemes in terms of two currency or three currency exchanges
under (2). Of course it is recognized that more complicated
possibilities might be constructed. In this section we investigate
the structural relations involved in n—country pure exchange net—
works. Our results will be segregated into those of interest to
the arbitrager, who is assumed to be unable as an individual to
affect the exchange rates by his own transactions, and those of
interest to an international monetary conference, which is con—
cerned with multi—lateral exchange rate agreements.

1.1.1 Arbitrage. We assert in the following statement that in
a complete pure exchange network the value of any chain remains
unaltered if a new currency is introduced within the chain as
shown below.

PROPOSITION 1: In a complete pure exchange network,

$$(3) \qquad x_1 x^1 \ldots x^p x_j x^1 \ldots x^q x_k = x_1 x^1 \ldots x^p x_j X_m \cdot X_m x_j x^1 \ldots x^q x_k .$$

Proof:

$$x_1 x^1 \ldots x^p x_j x^1 \ldots x^q x_k = x_1 x^1 \ldots x^p x_j \cdot x_j x^1 \ldots x^q x_k$$

$$= x_1 x^1 \ldots x^p x_j \cdot x_j X_m x_j \cdot x_j x^1 \ldots x^q x_k$$

$$= x_1 x^1 \ldots x^p x_j X_m \cdot X_m x_j x^1 \ldots x^q x_k$$

by applying in order, (1) (2) and then (1) three times.

Next we assert that exchanging currency in "different directions" along a chain in a pure exchange network determines reciprocal numbers.

PROPOSITION 2: In a pure exchange network,

$$x_1 x^1 x^2 \ldots x^{p-1} x^p x_j \cdot x_j x^p x^{p-1} \ldots x^2 x^1 x_1 = 1$$

Proof: This follows easily by applying (2),(1) and induction.

As a corollary we have for the case $j = i$ that in a pure exchange network the values derived from traveling in opposite directions around a cycle are reciprocal; consequently, if it is unprofitable to conduct a given "cycle" of exchange transactions in a pure exchange network, it is profitable to conduct the transactions in exactly the reverse order.

The next proposition states the fundamental property of complete pure exchange networks. If it is possible to arbitrage in the network, then every currency will be able to arbitrage in some three-way (triangular) transaction. Contrapositively, if triangular arbitrage does not exist for any given country X no arbitrage can exist in the entire network.

PROPOSITION 3: Let X_0 be any designated country in a complete pure exchange network, S. S is not in equilibrium if and only if it is possible to arbitrage from X_0 through two other countries (i.e. $X_0 X_1 X_j X_0 > 1$ for some i, j) .

Proof: The condition is obviously sufficient. To show that it is necessary assume that arbitrage is possible and thus that $X_a X_b X_c X_d X^1 \ldots X^p X_a > 1$ for the indicated countries. By application of (1), (2) and Proposition 1 we shall systematically turn the cycle into a series of triangular (4 letter) cycles from X_0. This is done in such a way that the product of the values of these cycles will be equal to the value of the original cycle and hence greater than one. Since a product of non—negative numbers can not be greater than one unless at least one of the factors is greater than one, we shall conclude that X_0 can arbitrage in at least one triangle. The construction follows. The reader may verify that at each step of the construction the value of the cycle or product of values of the cycles remains fixed.

Case 1.   a $\neq$ 0. Construct

$$X_0 X_a X_b X_c X_d X^1 \ldots X^p X_a X_0$$

using (1) and (2). Then consider the third and fourth letters of the cycle, X_b and X_c , and act as follows:

A. b $\neq$ 0, c $\neq$ 0. Construct

$$X_0 X_a X_b X_0 \cdot X_0 X_c X_d X^1 \ldots X^p X_a X_0$$

using Proposition 1.

B.   b - 0, c $\neq$ 0. Construct

$$X_0 X_a X_0 \cdot X_0 X_c X_d X^1 \dots X^p X_a X_0 - X_0 X_c X_d X^1 \dots X^p X_a X_c$$

using (1) and (2).

C.   b $\neq$ 0, c - 0. Construct

$$X_0 X_a X_b X_0 \cdot X_0 X_d X^1 \dots X^p X_a X_b$$

using (1).

Case 2. a - 0. In this case the cycle is of the form

$$X_0 X_b X_c X_d X^1 \dots X^p X_0 .$$

Perform the steps A, B, and C letting X_c and X_d of this cycle correspond respectively to the countries X_b and X_c of the other case.

Having performed these steps on the original cycle we now continue this process working from left to right on the remaining cycle if it has more than four letters. This cycle will satisfy case 2. We preserve untouched the four letter cycles produced (in alternatives A or C). Since three letter cycles correspond to factors of 1 which can be ignored, and because cycles (which are the only entities turned up by the algorithm) must have at least three letters, this method must result in a set of cycles of four letters or more.

However except possibly for the original application of the algorithm, the size of the cycle being dealt with must decrease by at least one letter. Hence the method must terminate in a set of four letter cycles. This, taken with the introductory argument, establishes the result.

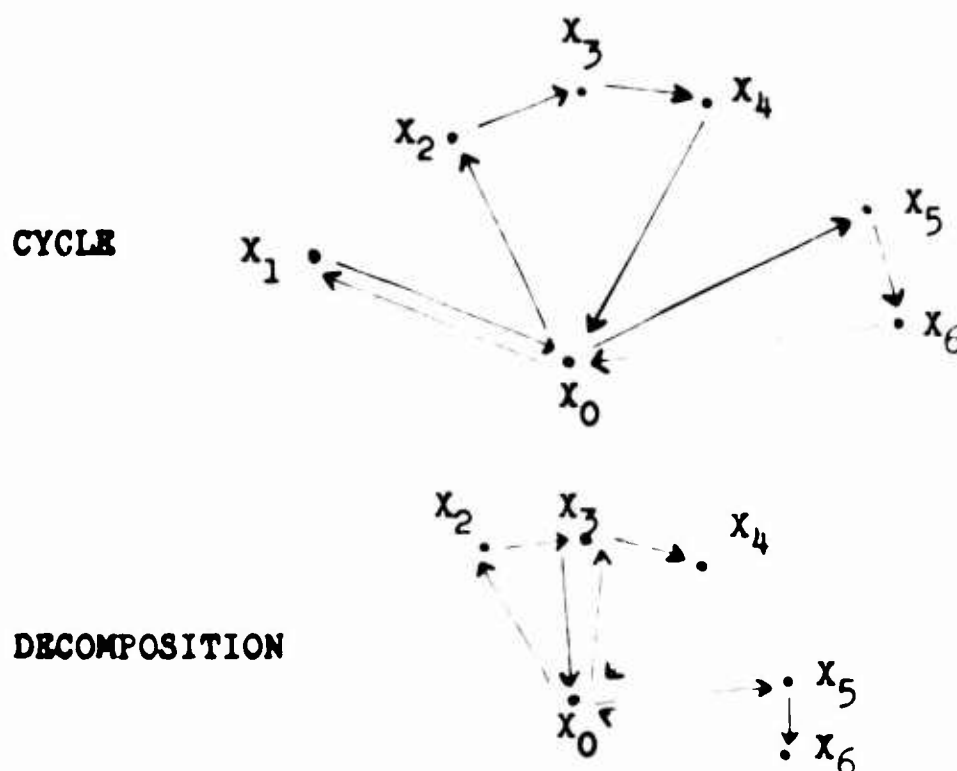Fig. 1 illustrates the decomposition in the proof for a possible 7 country cycle.

Fig. 1

As a corollary to this result we have

Corollary 1. Given a complete pure exchange network with n countries, it is sufficient to examine $\frac{(n-1)(n-2)}{2}$ cycles to determine whether or not arbitrage exists anywhere in the system. (This represents the number of triangles from one country.)

One might ask whether it is possible to establish definitely the existence of equilibrium by testing less than $(n-1)(n-2)/2$ cycles. The next result show that this is not possible, no matter how complex the cycles considered.

PROPOSITION 4: Given a complete pure exchange network with n countries, it is not possible to determine that the system is in equilibrium by examining fewer than $\frac{(n-1)(n-2)}{2}$ cycles.

Proof: Order the triangles through some fixed country X_0 and let y_i be the value of the ith triangle in some fixed direction, $i = 1,2,\ldots, \frac{(n-1)(n-2)}{2}$. We first show that the $\{y_i\}$ are independent (i.e. there is a network corresponding to any positive set $\{y_i\}$) for let $\{y_i^0\}$ be any desired positive set of values. Let $X_0 X_j$ take on any fixed positive values. Define

$$ X_{i_1} X_{i_2} = y_i^0 \cdot X_{i_1} X_0 \cdot X_0 X_{i_2} $$

where X_{i_1} and X_{i_2} are the countries involved in the ith triangle. Since the rate $X_{i_1} X_{i_2}$ appears in only the ith triangle this uniquely defines it and we can easily see that it gives the required value for y_i^0 . Hence for any set of positive values $\{y_i\}$ $i = 1,2,\ldots, \frac{(n-1)(n-2)}{2}$, there is a corresponding network. We shall show that for any set of cycles C_k , $k = 1,2,\ldots,k_0$, $k_0 < \frac{(n-1)(n-2)}{2}$ (and corresponding values)

there is a network which is not in equilibrium but which has rates such that the given cycles have the given values. Hence the cycles could not have determined that the given network was in equilibrium. By Proposition 3 each given cycle can be replaced by a set of triangular cycles with origin X_0, the product of which gives the value of the original cycle which we may take to be one.[*] Hence the given cycles correspond to the equations

$$\prod_{i=1}^{\frac{(n-1)(n-2)}{2}} y_i^{a_{ik}} = 1 \quad k = 1,2,\ldots,k_0$$

where the a_{ik} are integral exponents. Since $k_0 < \frac{(n-1)(n-2)}{2}$ it is evident that we may solve for k_0 of the y_i in terms of the other $\frac{(n-1)(n-2)}{2} - k_0$ variables. Hence at least one of the y_i may be chosen not to be equal to one. Allowing the others to be determined or chosen in any fashion consistent with the equations and the general positivity requirement, we determine a set of $\{y_i\}$. By the independence argument there exists a network with these y_i. By the construction of the $\{y_i\}$ the given cycles will have value one in this network. But since at least one triangle arbitrages in the network, it is not in equilibrium. Hence that the C_k, $k = 1,2,\ldots,k_0$ have value one is not sufficient to determine that the network is in equilibrium and the result is established.

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[*]If the cycle did not have value one, no network containing that cycle could be determined to be in equilibrium. Any country in the cycle could arbitrage by sending its currency through the cycle in the direction having its value greater than one (see Proposition 2.)

1.1.2 Multi-Lateral Adjustments   Up until now, we have con-
sidered relations that ascertain the existence of arbitrage
possibilities in a complete pure exchange network; the results
involved determing the values of a set of chains. Here we shift
attention to the component rates themselves. We establish con-
ditions which must exist among the exchange rates in order for
us to construct an equilibrium system, or to restore a system
to equilibrium; also we delineate the effects of exchange rate
alterations and devaluations upon a pure exchange system.

PROPOSITION 5: Given an incomplete pure exchange network in
equilibrium, S, which contains all n countries, there is a unique
complete pure exchange network S' , in equilibrium which contains
it.

Proof: Assume $X_i X_j$ is not defined in S. Since, by hypothesis,
all countries are part of S, there exists a path $X_i X^1 \ldots X^p X_j$
in S. Define $X_i X_j = X_i X^1 \ldots X^p X_j$. This definition is unique.
For assume that another path exists $X_i X^{p+1} \ldots X^{p+q} X_j$ in S.
By assumption S is consistent, so

$$X_i X^1 \ldots X^p X_j X^{p+q} \ldots X^{p+1} X_i = X_i X^1 \ldots X^p X_j \cdot X_j X^{p+q} \ldots X^{p+1} X_i = 1.$$

By proposition 2

$$X_i X^{p+1} \ldots X^{p+q} X_j \cdot X_j X^{p+q} \ldots X^{p+1} X_i = 1$$

and since no rates are zero we have

$$X_1 X^{p+1} \ldots X^{p+q} X_j \;-\; X_1 X^1 \ldots X^p X_j \;.$$

The network S with this rate adjoined is again consistent, for if arbitrage involving the new arc is possible, i.e.,

$$X_m X^1 \ldots X^s X_1 X_j X^{s+1} \ldots X^{s+t} X_m \;>\; 1,$$

then

$$X_m X^1 \ldots X^s X_1 X^1 \ldots X^p X_j X^{s+1} \ldots X^{s+t} X_m \;>\; 1$$

is easily seen to hold. But since this cycle involves only arcs in S, we have achieved a contradiction. Thus the assigning of rates can be carried out in such a way as to preserve consistency, and hence the subnetwork can be expanded into a consistent one. It is unique by construction, which gives the result.

As simple corollaries we have

Corollary 1: The n-1 rates between a currency and all other currencies in a pure exchange network determine a unique equilibrium in the complete network.

Corollary 2: A chain of links through all nodes, of the form $X_1 X_2 X_3 \ldots X_{n-1} X_n$, in a pure exchange network determines a unique equilibrium in the complete network.[*]

In our discussion to this point, we have made no assumption about the institutional aspects of currency markets; we have assumed only that a pure exchange network exists. The following theorems are directed at the implications of currency devaluation and alteration of rates. Consequently it is helpful to construct a hypothetical institutional framework in which such changes can be made. We suppose that each currency is managed by a central authority in the corresponding country. The authority has discretionary power to set exchange rates; but, since throughout this section we preserve (2), any direct alteration in the rate $X_1 X_j$ by X_1 is automatically agreed to by X_j in that X_j makes the reciprocal change in $X_j X_1$.

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[*]This result implies an alternative to Proposition 3, viz., in a complete pure exchange network, S, consisting of n countries, S is not in equilibrium if and only if one of the following equalities does not hold:

$$X_1 X_3 = X_1 X_2 X_3$$
$$X_1 X_4 = X_1 X_2 X_3 X_4$$
$$\vdots \qquad \vdots$$
$$X_1 X_n = X_1 X_2 X_3 \ldots X_n$$
$$X_2 X_4 = X_2 X_3 X_4$$
$$X_2 X_5 = X_2 X_3 X_4 X_5$$
$$\vdots \qquad \vdots$$
$$X_{n-2} X_n = X_{n-2} X_{n-1} X_n \quad .$$

We assert a well known statement that an across-the-board percentage change in any one country's rates (and the corresponding reciprocal change in the cross rates) does not create arbitrage possibilities.

PROPOSITION 6: If a currency is devalued, in a network in equilibrium, the network remains in equilibrium.

Proof: Let X_0 be the devaluer. Let a be the factor of devaluation. By Proposition 3 it is sufficient to verify that every triangle through X_0 has value 1. This verification is left to the reader.

Suppose that in p countries, the monetary authorities decide to change several or all of their exchange rates with other countries; what is the effect on the system? We first prove

PROPOSITION 7: Let S be a complete network in equilibrium. Let X_1, X_2,...,X_p, p < n, countries devalue thei. currency. Then the resulting network S' is independent of the order in which devaluation takes place.

Proof: Let primes indicate the new rates (in S') and consider a country X_s, s > p, which is not devaluing. Then defining k_1 by

$$X_s' X_1' = k_1 X_s X_1 \qquad 1 = 1, 2, \ldots p$$

and considering

$$X_s' X_1' = X_s X_1 \qquad 1 = p+1, \ldots, n \qquad 1 \neq s$$

we see that the rates attached to X_s are defined independently

of the order of devaluation. Since by Corollary 1 of Proposition 5
these rates uniquely determine S', S' is independent of the
order of devaluation.

We prove a statement which contains as a special case a
converse of Proposition 6.

PROPOSITION 8: Let S be a complete network in equilibrium.
Let S' be a complete network in equilibrium which arises from
S through the alteration of exchange rates by a group of countries,
say $X_1, X_2, \ldots, X_p$, $p < n$. Then S' is equivalent to a sys-
tem resulting from a uniquely determined devaluation of each of
the p currencies.

Proof: Let primes indicate the new rates (in S') and, considering
a currency X_s, $s > p$, which is not in the group, define the k_i
by the equations

$$X_s'X_i' = k_i X_s X_i \qquad i = 1,2,\ldots,p.$$

Evidently we also have

$$X_s'X_i' = X_s X_i \qquad i = p+1, p+2,\ldots,n \qquad i \ne s.$$

A complete network with these new rates for X_s could be achieved
by having each currency X_i, $i = 1,2,\ldots,p$, devalued by a factor
$1/k_i$, $i = 1,2,\ldots,p$. By Proposition 6 such a sequence of

devaluations would preserve equilibrium. By Corollary 1 of Proposition 5 there is a unique complete network in equilibrium with these rates. Hence this network must be S', and S' has been attained by devaluing each of the p currencies. The devaluation required of each currency is uniquely determined by k_i . This completes the proof.

A corollary is that if a single country alters its exchange rates by any process other than devaluation, arbitrage possibilities arise.

Corollary 1: The only unilateral action which preserves equilibrium is devaluation.

Furthermore by repeating the proof of Proposition 8 as if X_s wished to change p of its rates we have

Corollary 2: Let S be a complete network in equilibrium and let X_s change only p of its rates, $p < n$. Then to reach the new equilibrium determined by this action requires the equivalent of a devaluation of p currencies. The devaluations are uniquely determined.

Corollary 3: Let S, S' be complete networks in equilibrium. Then S may be transformed into S' by no more than n-1 devaluations.

We turn from discussing rate alterations which preserve equilibrium to the problem of establishing equilibrium in a given system by adjusting as few rates as possible. Such a question

might arise if there existed an international monetary conference which desired a good way of altering the current rates to remove the extant arbitrage possibilities.[*] We start by examining a case in which only relatively few rates need to be changed; this case includes the event in which only one rate is improperly set (i.e., in which a single alteration establishes equilibrium).

We define a complete network to be in near-equilibrium if it can be changed to an equilibrium network by altering fewer than $(n-1)/2$ rates. We demonstrate the relationship between a network in near-equilibrium and the triangular arbitrage transactions existing in the system.

PROPOSITION 9: A necessary and sufficient condition that a complete network, S, be in near-equilibrium is that

$$N(J) - \min_i N(i) < (n-1)/2 \quad ,$$

where $N(i)$ is the number of triangular arbitrage possibilities from X_i.

Proof: Assume $N(J) < \frac{n-1}{2}$. Let $X_j X_k X_m X_j$ permit arbitrage. Define $X_k' X_m' - X_k X_j \cdot X_j X_m$ to be the new rate replacing $X_k X_m$. Repeating this procedure for each instance of triangular arbitrage and leaving the rest of the network untouched, we create a new

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[*]The reader should note that in choosing among types of possible alterations which would restore a system to equilibrium, one would not necessarily select that set which requires the least number of altered rates. A full discussion of other alternatives is beyond the scope of this paper.

system S' in which X_J cannot arbitrage in any triangle. By
Proposition 3, S' is in equilibrium. Since only $N(J)$ arcs were
changed, S was in near-equilibrium and the condition is suf-
ficient. Now assume that the condition is not satisfied. Then
$N(i) \geq \frac{n-1}{2}$ for all i and because the equilibrium, S_i ,
determined by the ith country's rates (Proposition 5, Corollary 1),
can be achieved only by the construction above (changing a specific
rate in each of $N(i)$ triangles), S_i cannot be achieved by changing
fewer than $N(i) \geq \frac{n-1}{2}$ rates. Any other attempt to achieve
equilibrium must change at least one arc attached to every coun-
try, or at least $n-1$ arcs. Since $n-1 > \frac{n-1}{2}$ for all positive
n, S could not have been in near-equilibrium. This completes
the proof.

We next show the uniqueness of the equilibrium reached by
changing less than $(n-1)/2$ rates. We define $R(S,T)$, where
S and T are complete networks, to be the number of arcs which
are not the same in the two networks. R is a metric. In this
case the triangle inequality states that it requires at least
as many arc changes to transform network S_1 to S_2 and then
to S_3 as it would to transform S_1 to S_3 .

PROPOSITION 10: If S is in near-equilibrium, there is a unique
equilibrium situation S' such that $R(S,S') < \frac{n-1}{2}$.

Proof: By the previous assertion we know that an S_J exists
with the property in question. Assume that a **distinct** equilibrium
situation S' has the same property. Since the n-1 rates of
X_J uniquely determine an equilibrium situation (Proposition 5,

Corollary 1) it must be that $X_j' X_i' \neq X_j X_i$ for some i, where ' indicates the arcs in S'. By Corollary 2 of Proposition 8, the number of values, i, for which the inequality holds indicates the number of devaluations that must occur to take S into S'. Each devaluation changes n-1 arcs and no set of them can change fewer than n-1 . Hence $R(S_j, S') \geq n-1$. However since both S_j and S' have the property that their "distance" from S is less than $\frac{n-1}{2}$, we conclude by the triangle inequality,

$$R(S_j, S') \leq R(S_j, S) + R(S, S') < \frac{n-1}{2} + \frac{n-1}{2} = n-1$$

Hence it is a contradiction to say that S' and S are distinct, and the proposition is proved.

Propositions 9 and 10 enable one to identify the near-equilibrium networks and to use the constructive method in Proposition 9 with the certainty that it achieves equilibrium while changing the fewest number of rates.

These results permit one to recognize and repair the special case in which a previous equilibrium has been destroyed by a single incorrectly set rate (for n > 3). If the network is not in equilibrium and does not satisfy the condition of the previous proposition then the determination of an equilibrium system requiring the least number of altered rates is, in general, a difficult problem, and we defer a solution until Section 2.

1.2   GENERAL EXCHANGE NETWORKS

In this section we drop the assumption that the network is
a complete pure exchange currency network and consider general
networks.   In particular we remove (2) (and thereby the general
validity of Proposition 2).   We permit both $X_1 X_j X_1 > 1$ and
$X_1 X_j X_1 < 1$; the former possibility immediately leads to arbitrage,
and the latter possibility corresponds to admitting drains which
might include brokerage charges, currency shipment charges, and
insurance (insofar as these can be represented as a percentage
of the amount of currency exchanged).   Thus we permit the actual
or quoted rate to be diminished to reflect the existence of
ad valorem transaction costs.   These deviations from (2) are
usually of small magnitude and for that reason, several of the
previous results, e.g., those concerning devaluation, retain a
certain validity, although in a strict sense the assertions can
no longer be proved.   For the arbitrager, however, tiny deviations
from the pure exchange case are very important since, in reality,
his rate of profit is itself usually small; we postpone until
Sections 2 and 3 techniques of analysis open to the arbitrager
in the general network.   The crucial implication of relaxing
(2) is that Proposition 3 fails to hold.   We also relax the
assumption that the network need be complete.   In the general
network, the analogue of Proposition 3 is Proposition 3'.

PROPOSITION 3':   (1) Let X_0 be any country in a complete exchange
network, S, which satisfies $X_0 X_1 X_0 = 1$ for all 1.   Then it is

possible to arbitrage in S if and only if it is possible to arbitrage from X_0 through two other countries (i.e. $X_0 X_j X_k X_0 > 1$ for some (j,k)).

(11) In a general network, if it is possible to arbitrage, then it is possible to arbitrage in a series of exchanges which involves no more than one buying and one selling transaction for each country.

As the reader may verify, the hypotheses in (1) are the only conditions necessary to apply the construction of Proposition 3. The result is applicable if an arbitrager faces no transaction charges when dealing in his domestic currency; the effective charges would then have the necessary property.

The assertion of (11) follows from the fact that, even in a general exchange network, a cycle that passes through a country twice can be presented as two cycles whose product has the original cycle's value. Thus, applying (1) twice, we have

$$x_1 x^1 \ldots x^p x_j x^{p+1} \ldots x^q x_j x^{q+1} \ldots x^n x_1 =$$

$$x_1 x^1 \ldots x^p x_j \cdot x_j x^{p+1} \ldots x^q x_j \cdot x_j x^{q+1} \ldots x^n x_1 =$$

$$x_1 x^1 \ldots x^p x_j x^{q+1} \ldots x^n x_1 \cdot x_j x^{p+1} \ldots x^q x_j .$$

If the original cycle arbitraged, one of the derived ones must. The proposition follows by repeating the process as many times as is necessary.

The reader should observe that since the network is connected, if profits can be made by arbitraging, they can be exchanged into any other currency.

Proposition 4 remains true if the word "pure" is deleted, but it loses much of its significance unless a situation like the one hypothesized in (1) above exists. Proposition 5 and its corollaries remain correct if the words "pure" and "unique" are deleted. Propositions o and 7 hold[*] but Proposition 8 and its three corollaries do not; however if the deviations from pure exchange are small (as they usually are) and if the multilateral rate alterations contemplated are large (as they usually are) the results are essentially correct. Propositions 9 and 10 are no longer true and their significance remains unimpaired only in the case that rates are so badly set that arbitrage is of a different order of magnitude from that of the deviations from pure exchange (e.g. if rates are actually set at a distinctly incorrect level); as we show in Section 2, the problem with which these propositions are concerned is completely solved by a linear programming formulation.

1.2.1 Multi-Bloc Exchange. An incomplete network of particular interest is one in which the n countries are divided into m currency blocs, each of which has a single distinguished currency. Within each bloc a complete network exists and the network between the distinguished currencies is complete; but to exchange

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[*]We assume, as before, that if X_k changes his rate $X_k X_1$ to $k_1 X_k X_1$, then X_1 changes $X_1 X_k$ to $\frac{1}{k_1} X_1 X_k$.

a minor currency in one bloc for a minor currency in another
bloc, it is necessary to go through intermediary exchanges with
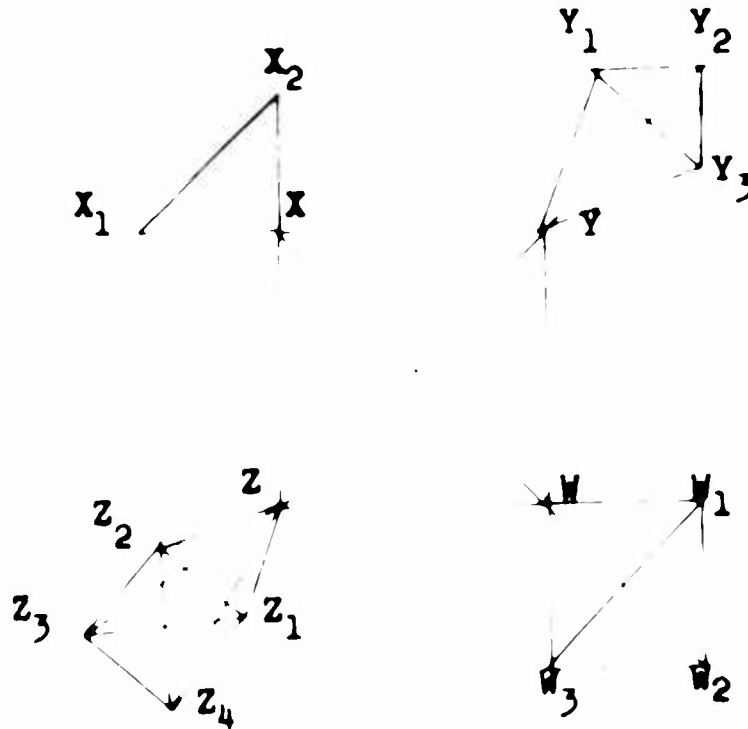the distinguished currencies. Such a network appears in Fig. 2.

Fig. 2

The "capital" countries here are X, Y, Z, and W. A typical
cycle might appear as below:

$$x_1 x^1 \ldots x^p x X Y^1 \ldots Y^q Y Z z^1 \ldots z^r z W W^1 \ldots W^s W X X_1$$

By successive application of (1) the reader can verify that
the following product of cycles can be substituted for it:

$$x_1 x^1 \ldots x^p X X_1 \cdot Y Y^1 \ldots Y^q Y \cdot Z z^1 \ldots z^r Z \cdot W W^1 \ldots W^s W \cdot X Y Z W X$$

If the original cycle had a value greater than one we again reason that one of the above cycles must. Hence it is sufficient (and obviously necessary) to examine, for the possibility of arbitrage, each of the complete networks which arises by considering a single bloc plus the complete network among the "capitals."

1.2.2 **Commodity-Currency Exchange.** Finally we consider networks in which certain nodes correspond to currencies while others correspond to commodities at various locations in time or space. At each country, there are defined exchange rates for commodities in terms of that country's currency, and an exchange rate for a type of commodity in one country in terms of the same type of commodity in another country (such a rate takes into account *ad valorem* shipping or holding charges levied in the transferal of a commodity from one country to another). We are interested in determining, under various circumstances, the number of transactions necessary to permit arbitrage when dealing in a commodity. Because we are primarily interested in commodity arbitrage, we confine our attention to the case in which it is impossible to arbitrage in money alone (i.e. the sub-network composed of currency nodes is in equilibrium). Since the costs of shipping commodities are so much larger than the cost attached to money transactions, we first consider the situation where the currency sub-network is a pure exchange network.

PROPOSITION 11. In a combination currency—commodity network
for which the currency sub—network is a complete pure—exchange
network, if an arbitrage possibility utilizing commodity trans—
actions exists, then only one commodity shipment need be made.[*]

Proof: Let X_1 stand for the currency of country 1 and Y_1
and Z_1 for particular commodities purchasable in country 1. Then,
should the cycle shown in Fig. 3 permit arbitrage, by utilising
the pure exchange of currency, we could substitute for it the
two cycles $X_1 X_2 Y_2 Y_3 X_3 X_1 \cdot X_1 X_3 X_4 Z_4 Z_5 X_5 X_1$ whose product must
again be greater than one. Hence one of these must permit
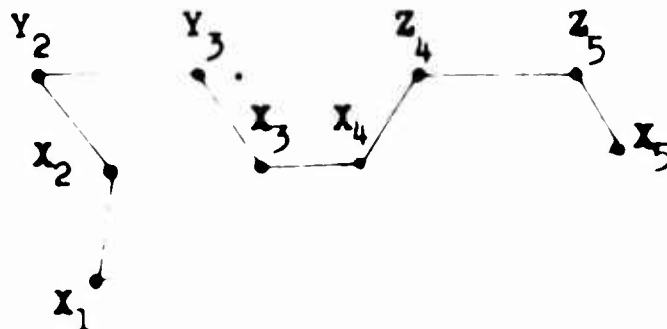arbitrage, while each involves only one commodity.

Fig. 3

One implication of the proposition is that under the
hypothesis, a firm of arbitragers may be sectioned into groups
of commodity specialists, each group searching for profitable
transactions by exchanging its own commodity. Such a division
of effort is bound to discover an arbitrage possibility if one
exists.

[*] In the case of futures trading, we interpret "shipment"
as the act of holding the commodity from one period of time to
another, while perhaps incurring storage costs; the transaction
might also involve physical transport of the commodity.

The proposition remains valid if the currency sub—network is not in equilibrium, but in this case it may occasionally be better to dispense with commodity transactions altogether. When we consider the situation where (2) does not hold in the currency sub—network, then the assertion is no longer true.

When the rate of exchange of currencies depends to a significant degree on the location of the currency (e.g. the dollar for pound rate in New York is not the same as the dollar for pound rate in London), the currency network becomes a special kind of commodity network. Unlike the commodity networks discussed thus far, it has no underlying currency sub—network. A diagram appropriate to the situation, Fig. 4 indicates several families of currencies, each attached to a location or home currency. A currency can change location only by exchange with home currencies or by shipment. In the latter case there may or may not be a cost attached to shipment. Dotted lines correspond to possibilities of shipment.
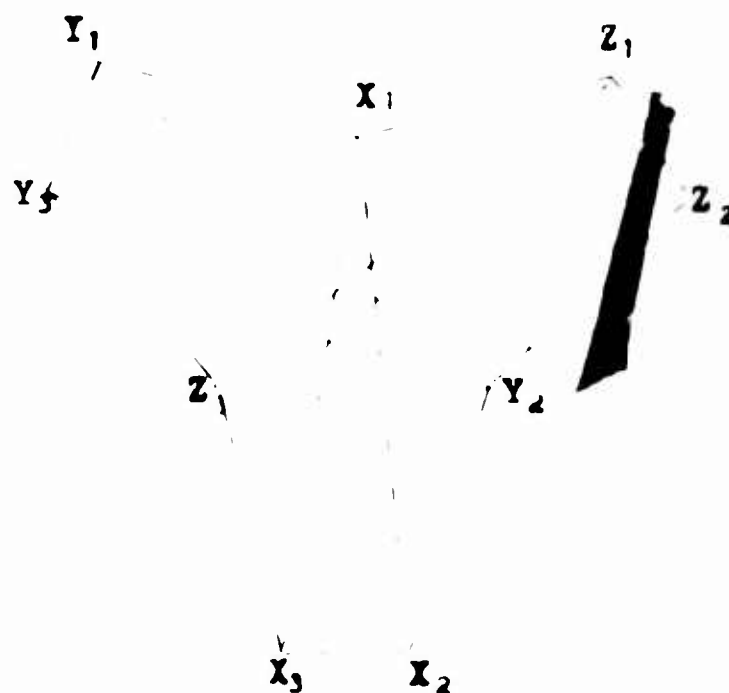
Y_1

Z_1

X_1

Y_3

Z_2

Z_1

Y_2

X_3 X_2

Fig. 4

2. A STATIC LINEAR PROGRAMMING MODEL

It is commonly recognized that linear programming may be
applied to several models of international trade [2, 5, 6, 8,
9, 10]. In this section we present a linear programming
formulation of exchange transactions which formally embodies
or extends several of the previous results. For the sake of
simplicity, we confine our discussion in this section to a
complete (not necessarily pure-exchange) system of currency
exchange; the reader will have no difficulty in making the
necessary modifications to allow for the possibilities of
commodity exchange or for an incomplete system.

We let $x_{1j} \geq 0$ be the number of units of currency X_1
exchanged for that of currency X_j and $c_{1j} > 0$ be the ex-
change rate of X_1's currency for X_j's currency.[*] To construct
the model which ascertains the existence of arbitrage, we let
$M > 0$ be an arbitrary amount of some currency, say X_1's, which
we wish to earn by means of currency transactions. The con-
straint equations of the linear programming model are

$$(4) \quad - \sum_{j=2}^{n} x_{1j} + \sum_{i=2}^{n} c_{11} x_{11} \geq M$$

and

$$(5) \quad - \sum_{j \neq k} x_{kj} + \sum_{j \neq k} c_{jk} x_{jk} \geq 0 \qquad k = 2, 3, \ldots, n .$$

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[*]Heretofore, we denoted c_{1j} by $X_1 X_j$.

The cost form being minimized is identically zero in this formulation. Equation (4) ensures that the net gain from all currency transactions involving X_1's currency is at least M, and Equation (5) ensures that the net gains in the remaining currencies are non-negative (i.e., no debts may be incurred). Note that if a feasible solution to (4) and (5) exists, any positive multiple (> 1) of all the x_{1j} also yields a feasible solution; equivalently, the feasibility of (4) and (5) is independent of M.*

If we are merely interested in locating an arbitrage scheme, only search for a feasible solution to the model. In addition we may wish the solution to have some "optimal" property. For example, we may desire to find that solution which involves the least amount of selling of X_1. In this case we would

$$(6) \qquad \min \sum_{j=2}^{n} x_{1j} \quad .$$

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*In the special case of a pure exchange network, i.e , $c_{1j} c_{j1} = 1$, we may reduce the number of variables by netting out x_{1j} and x_{j1}. Explicitly, let y_{1j}, $1 < j$, denote the net exchange of X_1's currency for that of X_j's

$$- y_{1j} = - x_{1j} + c_{j1} x_{j1}.$$

The variable y_{1j} is unrestricted in sign. We note that $c_{1j} y_{1j} = c_{1j} x_{1j} - x_{j1}$, and thus the model becomes

$$(4') \qquad - \sum_{j=2}^{n} y_{1j} \geq M$$

$$(5') \qquad - \sum_{j>k} y_{kj} + \sum_{j<k} c_{jk} \geq 0 \qquad \qquad k = 2,3, \ldots, n.$$

Or there may be a fixed transaction cost d_{ij} involved in making

the exchange x_{ij} , and we may wish to find a solution which

minimizes the sum of the transaction costs; the special case of

$d_{ij} = 1$ corresponds to minimizing the total number of transactions

used in the solution. In this case we add to the equations

$$(7) \qquad x_{ij} - L \, \gamma_{ij} \leq 0 \qquad \gamma_{ij} = 0 \text{ or } 1$$

and $L > 0$ is an arbitrarily large number

$$(8) \qquad \min \ \Sigma \ \gamma_{ij} d_{ij} \qquad .$$

Constraint (7) ensures that whenever $x_{ij} > 0$, $\gamma_{ij} = 1$,

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If we convert (4') and (5') to equalities by adding slack
variables r_k, k = 1, 2, ..., n, and utilize the fact that the
y_{ij} are unconstrained in sign, we can solve for y_{ij}, j = 2, 3,...,n,
(or alternatively $y_{j,j+1}$, j = 1, 2, ..., n-1) in (5') and eliminate
these variables by substitution in (4') yielding

$$\Sigma \ \alpha^{\bullet}_{ij} \ y^{\bullet}_{ij} - \sum_{k=1}^{n} \beta_k r_k = M \qquad \beta_k > 0$$

where the $y^{\bullet}_{ij}$ are the remaining variable and $\alpha^{\bullet}_{ij}$ their co-
efficients. A necessary and sufficient condition for no arbitrage
to be possible is that all the $\alpha^{\bullet}_{ij} = 0$. One can derive from the
relations $\alpha^{\bullet}_{ij} = 0$ either Proposition 3 or equivalent conditions
for equilibrium, depending on the y_{ij} chosen for elimination.
We also note that since we only need to investigate basic solu-
tions in a linear programming model, part (11) of Proposition 3'
can be verified from the programming model; in fact, if an
arbitrage possibility exists, it is possible to earn M units
of any currency in no more than n transactions, given the
assumptions of the static formulation.

and (8) ensures that $\gamma_{ij} = 1$ only if $x_{ij} > 0$. The system
with (7) added is a linear programming model involving several
integer—valued variables; consequently one must resort to an
algorithm such as that of R.E. Gomory [4] to solve the problem.

Despite the simplicity of structure and compactness of
constraints of (4) and (5), the model has two main drawbacks.
First, the arbitrage possibility indicated by the model may
require that the arbitrager is able to execute his exchange
transactions instantaneously (i.e. without liquidity con-
straint) or, alternatively, that he possess capital reserves
in some other country than his own. An example of such a
solution would be an American arbitrager earning dollars by
selling pounds for francs, francs for lire, and lire for pounds,
and then converting his profits in pounds into dollars. Such
a chain of transactions might be impossible unless the arbitrager
had a certain amount of foreign (pounds) funds available to
"pump—prime" the pounds—francs—lire cycle, or were in a position
to act instantaneously.* Second, the model is ill—equipped to
solve the problem of finding arbitrage possibilities in a few
transactions. Although the technique illustrated in (4) and
(5) can be adapted trivially to provide an upper bound constraint
on the number of allowable transactions (number of non-zero x_{ij}),
the complexity of the partial integer linear programming problem
is increased considerably. The model presented in the next

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*The arbitrager would also have the possibility of converting
dollars into pounds, compounding profits in the pound—franc—lire
market by repeating the arbitrage cycle, discovered by the model,
provided the profitable set of rates existed long enough, and then
converting enough pounds back to dollars to overcome an "un-
favorable" dollars—to—pounds conversion rate.

section, although it apparently involves more equations, over-
comes these difficulties and also has a special structure ena-
bling the direct computation of a complete solution without
resort to an iterative technique such as the simplex method.
Consequently we do not recommend the arbitrager use (4) and (5).

Our present model is well suited to solving the international
adjustment problem posed in Section 1.1.2, viz., given a system
S which is not in equilibrium, to find an equilibrium system S'
which involves changing the least number of rates in S. However
by Proposition 5 any consistent sub-system can be extended to
a complete equilibrium system, and hence our problem reduces to
finding the largest sub-system in equilibrium (the arcs which
are to remain the same must certainly be in equilibrium). Hence
we wish to find the largest sub-system such that (1) and (2)
are infeasible. Consider the dual problem: corresponding to
each x_{ij} we have

(6) $- v_i + c_{ij} v_j \leq 0$ $v_k \geq 0$ $\max M v_1$

where the v_k are the dual variables.

We know from the Dual Theorem [3] that no solution to (1)
and (2) exists only if either no solution or an infinite solution
exists for (6). But $v_k = 0$, for all k, is an obviously feasible
solution to (6); consequently to solve our problem we want to
find the maximal number of dual relations which admit an infinite
solution. We observe the relations (6) are homogeneous, and
thus an infinite solution exists if and only if a feasible solution

to the subsystem can be found with $v_1 = 1$ (because of the homo-geneity, the maximal subsystem is independent of the actual value chosen for v_1). Consider a related dual problem:

$$(7) \quad -v_1 + c_{1j} v_j - t_{1j} = 0 \qquad v_k \geq 0, \qquad t_{1j} \geq 0 \qquad v_1 = 1$$

$$(8) \quad t_{1j} - S \epsilon_{1j} \leq 0 \qquad \epsilon_{1j} = 0,1,$$

and $S > 0$ is a large positive number

$$(9) \quad \min \Sigma \epsilon_{1j} \quad .$$

Note that (8) ensures that $\epsilon_{1j} = 1$ if $t_{1j} > 0$, and (9) ensures that $\epsilon_{1j} > 0$ only if $t_{1j} > 0$. The model solves for a set of v_k ($v_1 = 1$) which exactly satisfies as many relations in (6) as possible; any relations in (7) where $t_{1j} > 0$ implies the original dual relation in (6) is not satisfied. Those relations for which $t_{1j} = 0$, indicate the maximal sub-network.

In the pure exchange network, all we need are

$$(7') \quad -v_1 + c_{1j}v_j + u_{1j} - w_{1j} = 0 \qquad 1 < j \quad v_k \geq 0, \; u_{1j} \geq 0, w_{1j} \geq 0$$
$$v_1 = 1$$

$$(8') \quad u_{1j} + w_{1j} - S\epsilon_{1j} \leq 0 \qquad 1 < j, \; \epsilon_{1j} = 0, 1,$$
and S is a large positive number
$$(9') \quad \min \sum \epsilon_{1j} \quad .$$

3. A NETWORK FLOW METHOD

Suppose that a potential arbitrager with resources in some country's currency, not necessarily "dollars", wishes to maximize his profit in "dollars" subject to the restriction that he enter into no more than n transactions. We will present a model which solves this problem and at the same time solves an identical problem for initial resources in the currency of any other country.

We first consider the problem of maximizing profit while restricting the number of transactions to n or less, where n is the number of currencies in the system under consideration. In order to visualize the possibilities of buying and selling that might arise, we construct a diagram for n - 3, such as in Fig.⁵

Figure 5

The c_{ij}'s act as flow amplifiers or deamplifiers and represent the exchange rates. The problem is to maximize the flow entering the sink on the bottom right (i.e. the number of units of X_3's currency.) Fortunately this sort of linear

programming problem can be solved in an exceedingly simple way,
as has been noticed by W. Prager [7] and utilized by A. Charnes,
W. W. Cooper, and M. Miller in [1]. If the equations which
describe "conservation" of flow for the various nodes are
written down in an obvious systematic way, taking the nodes
in the first transaction, then the nodes in the second trans-
action, etc., the matrix expressing the problem takes the form
in Fig. 6.* The x_{ij}^k represent the amounts of X_i's money turned
to X_j's at "stage k." The equations on page 35A correspond
to the example above.

The R_i i = 1, 2, 3, are the available resources at X_i.
The u_i, i = 1, 2, ..., 12, represent the dual variables. We
will solve the problem by assigning values to the dual variables
in such a way as to minimize $u_1 R_1 + u_2 R_2 + u_3 R_3$. This will
determine an optimal solution by the Dual Theorem [3]. In
order to solve the dual problem, we simply assign $u_{12} = 1$,
$u_{11} = c_{23}$, $u_{10} = c_{13}$. The dual equations then require that we
assign u_7, u_8 and u_9 so that

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*By adding the "flow in—flow out" equations for all nodes
corresponding to the same currency, we generate a set of new
equations. If the variables $y_{ij} = \sum_{k=1}^{3} x_{ij}^k$ are substituted in
these equations, the columns, or activities, will take the form
exhibited in the static model.

Fig. 6

$$u_9 \geq \max\ (u_{12}, c_{32}u_{11}, c_{31}u_{10}) = \max\ (1, c_{32}c_{23}, c_{31}c_{13})$$

$$(10)\quad u_8 \geq \max\ (c_{23}u_{12}, u_{11}, c_{21}u_{10}) = \max\ (c_{23}, c_{23}, c_{21}c_{13})$$

$$u_7 \geq \max\ (c_{13}u_{12}, c_{12}u_{11}, u_{10}) = \max\ (c_{13}, c_{12}c_{23}, c_{13})\ .$$

Having assigned values to u_9, u_8 and u_7 which satisfy (10), we derive similar equations for u_6, u_5 and u_4 and for u_3, u_2 and u_1 . In order to minimize $u_1 R_1 + u_2 R_2 + u_3 R_3$, we evidently must minimize u_1, u_2 and u_3 ($R_i \geq 0$, $i = 1,2,3$) and thus we can do no better than to define the dual variables to be as small as possible at each step of the procedure. Hence we determine u_9, u_8 and u_7 so that equality holds in (10) and continuing in this way we assign values to all the dual variables.[*] We obtain a solution to the original problem by throwing away the set of columns whose dot product with the dual variables is not exactly zero and solving this simplified system with the last three columns.

A little consideration will show that despite the large size which the matrix might present, e.g. in a case involving 50

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[*] A dynamic programming, or functional equation, approach provides an alternative means for deriving the algorithm. Let $f_k(X_i)$ denote the maximum amount of currency of country X_n achievable starting with a unit of currency of country X_i and making k exchange transactions. We allow for the foregoing of an exchange by defining $c_{11} = 1$ to be the exchange rate of a unit of currency of country X_i for itself. If a single transaction is permitted,
$$f_1(X_i) = c_{in} \qquad i = 1,2,\ldots,n.$$
The recursion relation is
$$f_{k+1}(X_i) = \text{Maximum}_j\ \left\{ c_{ij} f_k(X_j) \right\}\ ,$$
which is analogous to (10).

currencies, the exceedingly simple way of assigning dual variables will keep the problem manageable. Two further important observations should be made. Since the method of assigning dual variables is independent of the currency inputs, and since the value of the optimal solution can be computed by taking the dot product of the dual variables with the right hand side, it is apparent that the value of a dual variable indicates that extra amount of profit which might be realized by introducing, at the corresponding node, one unit of the appropriate currency. As a result, our solution of the problem for n stages has also solved the problem for all intermediate stages. When we reach any stage in which a dual variable is seen to exceed the corresponding rate of exchange, we have located a possibility of arbitrage, and we might use the given resources to take advantage of this possibility for arbitrage. However, one might prefer to consider more stages in the interest of locating greater possibilities of profit before a decision is reached. In adding stages, it is not necessary to begin again but only to start with the previously found dual variables, to add the new relations at the beginning, and to continue computing the values of dual variables until sufficiently many are assigned. Thus our initial decision to work with n stages, where n is the number of currencies, is not essential. n stages, however, have a certain significance in that they represent sufficiently many stages to indicate any existing arbitrage by (11) of Proposition 3'.

In setting up the network flow model as in all models of general exchange networks, a question arises as to exactly which rates should be considered as the rates of exchange. An arbitrager might feel that the real rate of exchange should reflect the brokerage charge or, perhaps, the interest foregone in actually making a transaction. From a mathematical point of view it might be desirable to adjust the real rates in such a way that they indicate an arbitrage possibility only in cases where it is profitable to arbitrage after costs and in those cases indicated the percentage of profit that could be made after broker's costs by a single trip around the cycle involved. This adjustment can be made by multiplying the given c_{ij}'s by $(1-p_{ij})$ where p_{ij} is the percentage taken by the broker (usually 1/32 or 1/64 of 1%) in a transaction from X_i to X_j .

If the arbitraging contemplated will take an appreciable time, the network flow model can be constrained to deal in present discounted funds and, thus, to indicate arbitrage only when the arbitrage presents a more profitable opportunity than leaving one's money to collect interest for the same amount of time. Let r_{ij} be the percentage of interest that could be collected during the time it takes to make a transaction from X_i to X_j . Let the adjusted rates c'_{ij} be defined by

$$c'_{ij} = \frac{c_{ij}}{(1+r_{ij})}$$

It can be easily seen that the flow has become one of present discounted money.

The network flow model can also be used as a method for considering opportunities for speculation. If different rates of exchange are expected to occur over time, these rates can be introduced in the appropriate stage and the stages thought of as representing time periods. A discount factor of the type above may be used as a first approximation to provide a "certainty equivalent" for the risk involved in the speculative activity.

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